Dynamical Donuts

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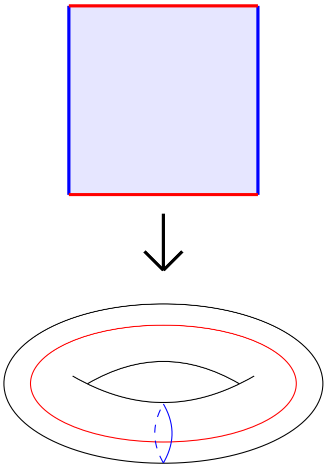
Table of Contents

Table of Contents

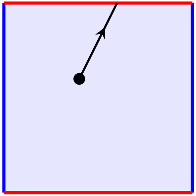
The Torus

- A square with sides identified by translations.
- We'll call it the translation torus.
- What can we study?
  - Topology
  - Geometry
  - Dynamics



Dynamics on a Translation Torus

- Natural straight-line flow.
- Flows on a torus live in two different dynamical worlds.
  - First: Periodic flows



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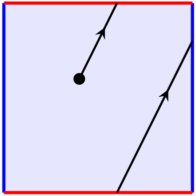

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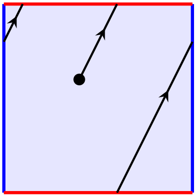

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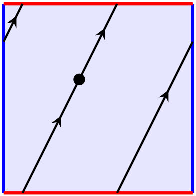

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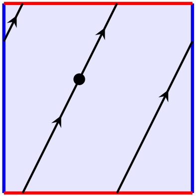
  - When?



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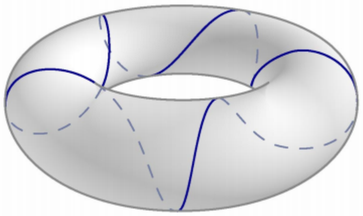

Dynamics on a Translation Torus

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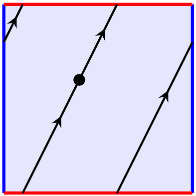
  - Rational slopes



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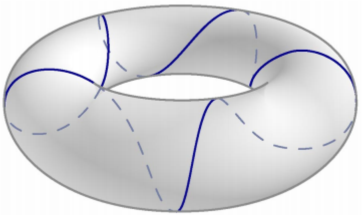

Proof: Rational Flows are Periodic

- Start at the point (x, y) on the translation torus $[0, 1]^2$.

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- $\{y + mq\} = \{y + p\} = \{y\}$, since $p \in \mathbb{Z}$.
- We are back at $(\{x + q\}, \{y + p\}) = (x, y)$ after time $t = q$. $\square$

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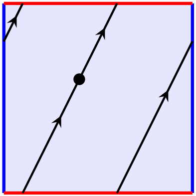

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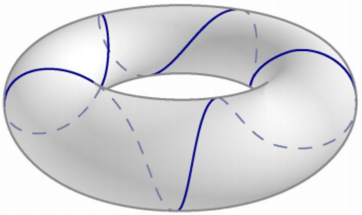
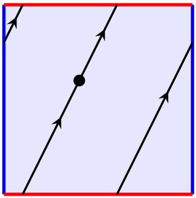


Dynamics on a Translation Torus

- Rational slopes: Periodic flows
- What about irrational slopes?
- ...



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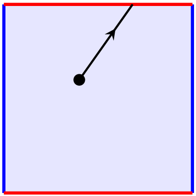

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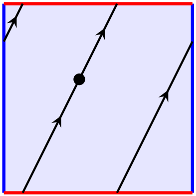

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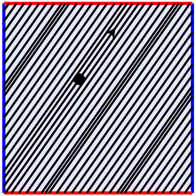

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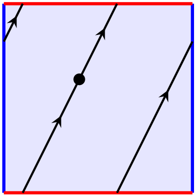
Dynamics on a Translation Torus

- Rational slopes: Periodic flows
- What about irrational slopes?
- The flow is dense!



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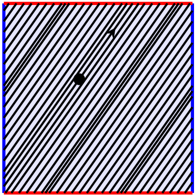

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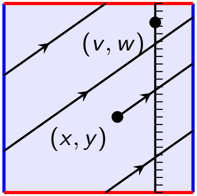
Proof Sketch: Irrational Flows are Dense



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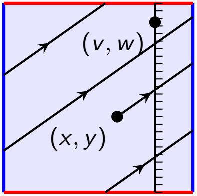

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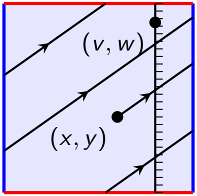

- Start at the point (x, y) on the translation torus $[0, 1]^2$.
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- Let (v, w) be a point, $\varepsilon > 0$. Do we get within ε of (v, w)?

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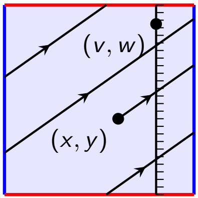

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- Flow until $\{x + t\} = v$.

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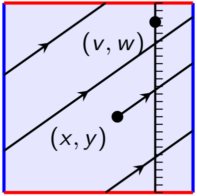

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- Split the vertical axis into boxes of size ε.

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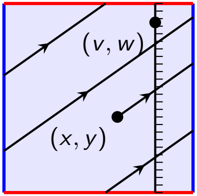

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- Let (v, w) be a point, $\varepsilon > 0$. Do we get within ε of (v, w)?
- Flow until $\{x + t\} = v$.
- Split the vertical axis into boxes of size ε.
- Every time you come back to $\{x + t\} = v$, mark a point on the vertical axis.

Proof Sketch: Irrational Flows are Dense

- Pigeonhole principle: Eventually, we mark two points in the same box: $\{y + (v - x) + mn_1\}$ and $\{y + (v - x) + mn_2\}$.

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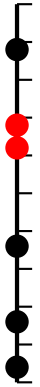

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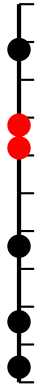

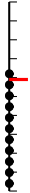

- Pigeonhole principle: Eventually, we mark two points in the same box: $\{y + (v - x) + mn_1\}$ and $\{y + (v - x) + mn_2\}$.
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- $\{m(n_2 - n_1)\}$ is within ε of 0.
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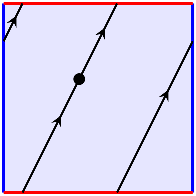
- ...eventually, $\{w - y - (v - x)\}$ is between $\{km(n_2 - n_1)\}$ and $\{(k + 1)m(n_2 - n_1)\}$, and so it's within ε of $\{km(n_2 - n_1)\}$.
- So $\{y + (v - x) + km(n_2 - n_1)\}$ is within ε of w! $\square$

Dynamics on a Translation Torus

- Rational slopes: Periodic flows
- What about irrational slopes?
- The flow is dense!



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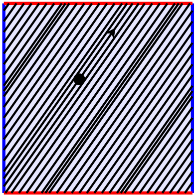

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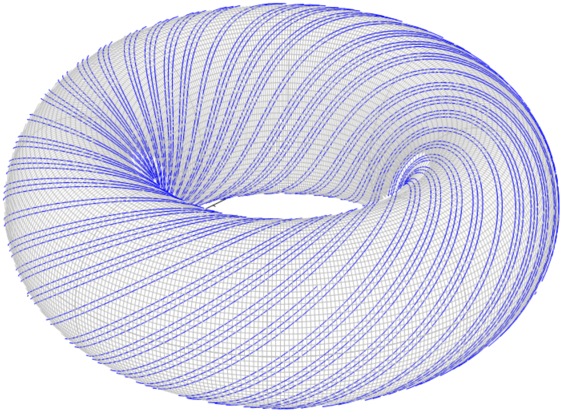
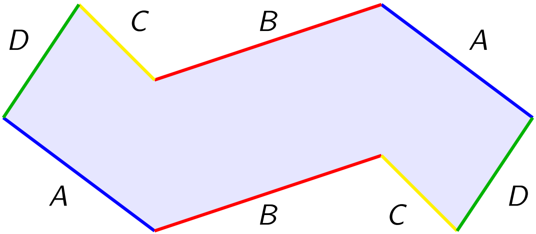


Table of Contents

Translation Surfaces: Construction

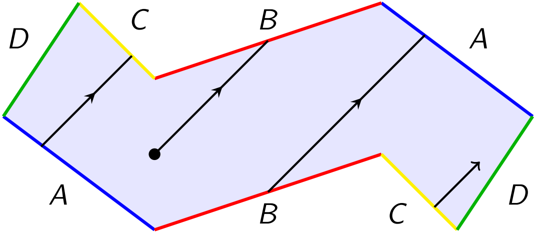
One way to define them:

- Start with a polygon.
- Identify pairs of sides by translations.
- Result: A surface with no curvature, finitely many cone points.



Translation Surfaces: Dynamics

No curvature: There is a natural straight-line flow.

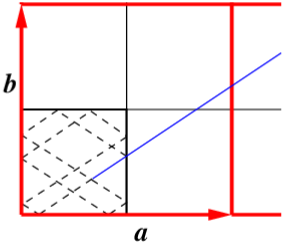


Theorem (S. Kerchoff, H. Masur, J. Smillie)

For any translation surface S the straight-line flow in almost any direction is dense.

Translation Surfaces: Applications

- Rational billiards (unfold to get a translation surface)
- Illuminating rooms and blocking sets

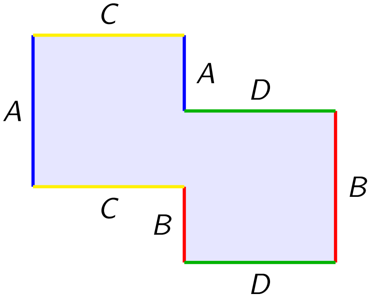


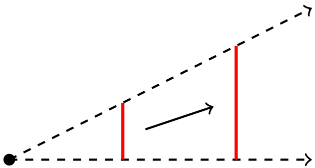
Avoiding a Laser
in a Translation Surface
(Nils Berglund)

Dilation Surfaces: Construction

Similarly to translation surfaces:

- Start with a polygon.
- Identify pairs of sides by translations or dilations.
- Result: A surface with no local curvature, but nontrivial holonomy (distances change when you go around loops).





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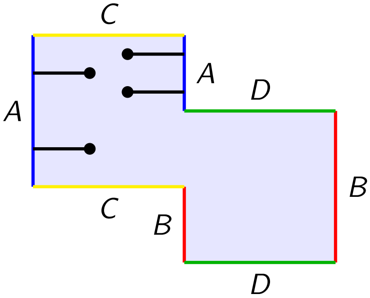
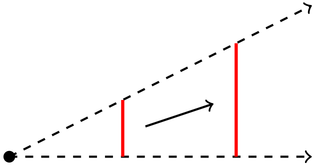




Dilation Surfaces: Dynamics

- There is still a natural straight-line flow.
- But now the dynamics are different!
- Here: The flow converges to a periodic orbit.

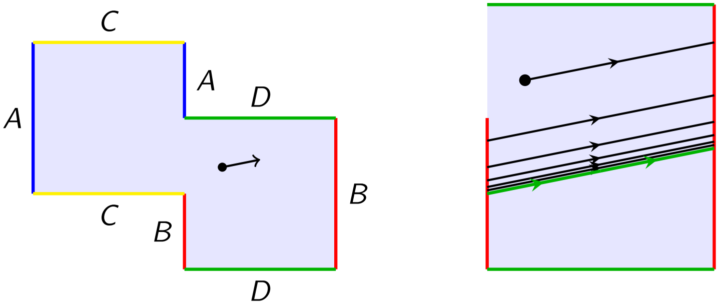
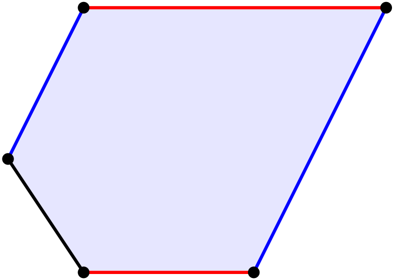
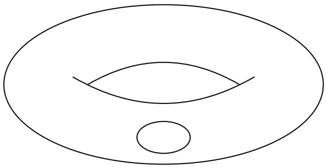


Table of Contents

Dilation Tori with a Single Boundary Component

- A simple class of dilation surfaces.
- One piece of boundary, one cone point.
- Folds together into a torus with a hole.



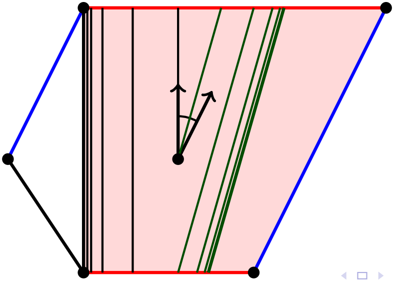


Dilation Tori: Dynamics

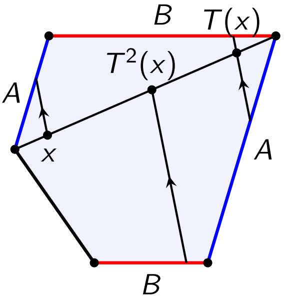
Consider the straight line flow in directions which point away from the boundary. We find the following result:

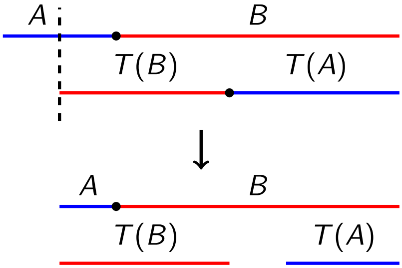
Theorem (H.-Wang)

Let S be a dilation torus a with single boundary component. The straight-line flow in almost any direction converges to a periodic orbit in S. The remaining directions form a Cantor set.



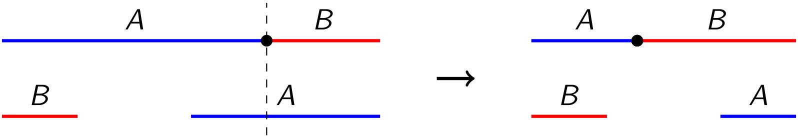
Proof Technique



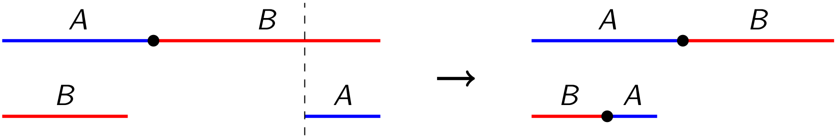


Restrict to a diagonal for an
affine interval exchange transformation

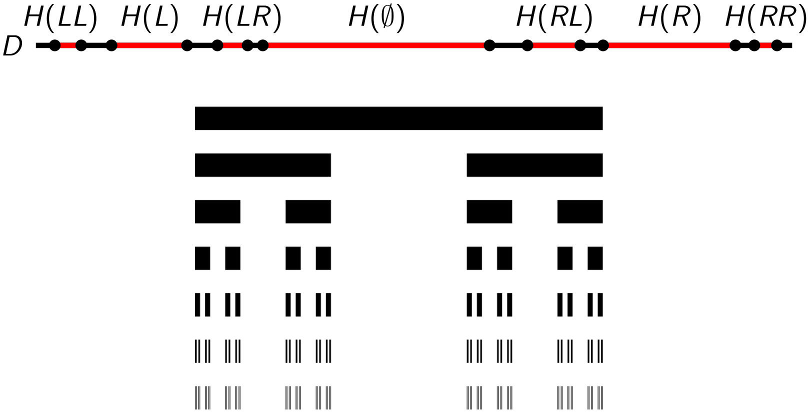
Proof Technique



Rauzy induction to move the breakpoint over the gap.



Proof Technique

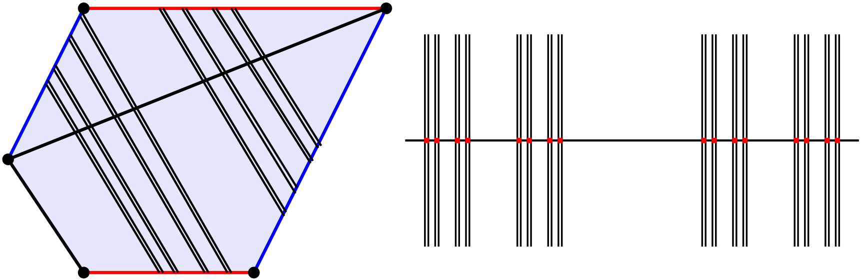


Dilation Tori: Cantor Sets

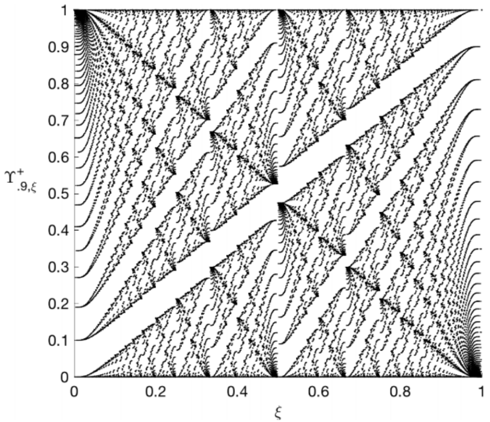
What happens to the remaining Cantor set of directions?

Theorem (H.-Wang)

Let C be the Cantor set of directions in S whose flow does not converge to a periodic orbit. For all but countably many directions in C, the flow accumulates to a set whose cross section is a measure zero Cantor set.



Dilation Tori: Limit Sets



ξ parameterizes direction. Plot by Bowman and Sanderson.

Next Steps

- Dilation surfaces have rich dynamical behavior.
- Torus represents translation surface dynamics.
- Do dilation tori represent dilation surface dynamics?

Conjecture (Ghazouani)

For any dilation surface S which is not a translation surface, the straight-line flow in almost any direction converges to some periodic orbit.

Questions

Thank you for listening.

Any questions or comments are welcome.

Proof Sketch of Kerchoff-Masur-Smilie

Moduli Space

To prove unique ergodicity, we shift the problem from focusing on a specific translation surface to considering the space of all translation surfaces $\mathcal{M}$.

On this space, we have a natural topology from nudging the sides of translation surfaces which gives it the structure of an orbifold.

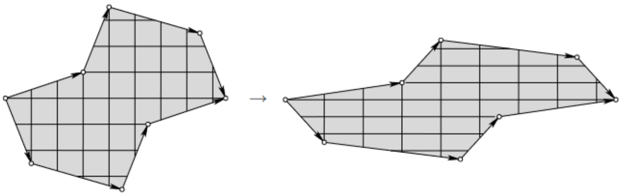
We have a natural $GL(2, \mathbb{R})$-action.

Teichmüller Flow

The $GL(2, \mathbb{R})$-action just applies a linear map to a translation surface as it is cut up on a plane. The most important subaction is the Teichmüller flow:

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

which renormalizes the vertical trajectory.



Proving Kirchoff-Masur-Smilie

Challenge: Find a probability measure on the moduli space of area-1 translation surfaces for which the Teichmuller flow is ergodic.

Achieved by Masur and Veech in 1982. Comes from an explicit density f_m on interval exchange maps.

$$f_m(\lambda) = \frac{1}{Z}\left(\frac{1}{1-\lambda_1} + \frac{1}{1-\lambda_m}\right)\prod_{i=1}^{m-1}\frac{1}{\lambda_i + \lambda_{i+1}}$$

Step 2: Use the ergodic flow to show that the vertical trajectory on any translation surface rotated to almost any angle is dense and equidistributed.

The Magic Wand

In 2013, Alex Eskin and Maryam Mirzakhani proved the celebrated "Magic Wand Theorem":

Theorem (A. Eskin, M. Mirzakhani)

The closure of any $GL(2,\mathbb{R})$-orbit in $\mathcal{M}$ is a complex suborbifold represented locally by affine subspaces. Any ergodic $SL(2,\mathbb{R})$-invariant measure is supported on a suborbifold.

Once you know the geometry of the $GL(2,\mathbb{R})$ orbit of a translation surface, you can answer all kinds of questions about it.